CHAPTER 14

CURVES

We all have a strong intuitive sense of what a curve is. Although we never see a curve floating around free of any object, we can readily identify the curved edges and silhouettes of objects and easily imagine the curve that describes the path of a moving object. This chapter explores the mathematical definition of a curve in a form that is very useful to geometric modeling and other computer graphics applications: that definition consists of a set of *parametric equations*. The mathematics of parametric equations is the basis for Bézier, NURBS, and Hermite curves. The curves discussed in this chapter may be placed in the Hermite family of curves. Bézier curves are the subject of the next chapter, and NURBS are best left for more advanced texts on geometric modeling. Both plane curves and space curves are introduced here, followed by discussions of the tangent vector, blending functions, conic curves, reparameterization, and continuity and composite curves.

14.1 Parametric Equations of a Curve

A parametric curve is one whose defining equations are given in terms of a single, common, independent variable called the *parametric variable*. We have already encountered parametric variables in earlier discussions of vectors, lines, and planes.

Imagine a curve in three-dimensional space. Each point on the curve has a unique set of coordinates: a specific x value, y value, and z value. Each coordinate is controlled by a separate parametric equation, whose general form looks like

$$x = x(u), \quad y = y(u), \quad z = z(u) \tag{14.1}$$

where $x(u)$ stands for some as yet unspecified function in which u is the independent variable; for example, $x(u) = au^2 + bu + c$, and similarly for $y(u)$ and $z(u)$. It is important to understand that each of these is an independent expression. This will become clear as we discuss specific examples later.

The dependent variables are the x, y, and z coordinates themselves, because their values depend on the value of the parametric variable u. Engineers and programmers who do geometric modeling usually prefer these kinds of expressions because the coordinates x, y, and z are independent of each other, and each is defined by its own parametric equation.

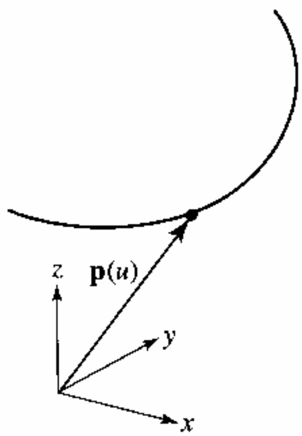


Figure 14.1 *Point on a curve defined by a vector.*

Each point on a curve is defined by a vector **p** (Figure 14.1). The components of this vector are $x(u)$, $y(u)$, and $z(u)$. We express this as

$$\mathbf{p} = \mathbf{p}(u) \tag{14.2}$$

which says that the vector **p** is a function of the parametric variable u.

There is a lot of information in Equation 14.2. When we expand it into component form, it becomes

$$\mathbf{p}(u) = [x(u) \quad y(u) \quad z(u)] \tag{14.3}$$

The specific functions that define the vector components of **p** determine the shape of the curve. In fact, this is one way to define a curve—by simply choosing or designing these mathematical functions. There are only a few simple rules that we must follow: 1) Define each component by a single, common parametric variable, and 2) make sure that each point on the curve corresponds to a unique value of the parametric variable. The last rule can be put the another way: Each value of the parametric variable must correspond to a unique point on the curve.

14.2 Plane Curves

To define plane curves, we use parametric functions that are second degree polynomials:

$$\begin{aligned} x(u) &= a_x u^2 + b_x u + c_x \\ y(u) &= a_y u^2 + b_y u + c_y \\ z(u) &= a_z u^2 + b_z u + c_z \end{aligned} \tag{14.4}$$

where the a, b, and c terms are constant coefficients.

We can combine $x(u)$, $y(u)$, $z(u)$, and their respective coefficients into an equivalent, more concise, vector equation:

$$\mathbf{p}(u) = \mathbf{a}u^2 + \mathbf{b}u + \mathbf{c} \tag{14.5}$$

We allow the parametric variable to take on values only in the interval $0 \leq u \leq 1$. This ensures that the equation produces a bounded line segment. The coefficients **a**, **b**, **c**, in this equation are vectors, and each has three components; for example, $\mathbf{a} = [a_x \quad a_y \quad a_z]$.

This curve has serious limitations. Although it can generate all the conic curves, or a close approximation to them, it cannot generate a curve with an inflection point, like an S-shaped curve, no matter what values we select for the coefficients **a**, **b**, **c**. To do this requires a cubic polynomial (Section 14.3).

How do we define a specific plane curve, one that we can display, with definite end points, and a precise orientation in space? First, note in Equation 14.4 or 14.5 that there are nine coefficients that we must determine: $a_x, b_x, \ldots, c_z$. If we know the two end points and an intermediate point on the curve, then we know nine quantities that we can express in terms of these coefficients (3 points × 3 coordinates each = 9 known quantities), and we can use these three points to define a unique curve (Figure 14.2). By applying some simple algebra to these relationships, we can rewrite Equation 14.5 in terms of the three points. To one of the two end points we assign $u = 0$, and to the other $u = 1$. To the intermediate point, we arbitrarily assign $u = 0.5$. We can write this points as

$$\begin{aligned} \mathbf{p}_0 &= [x_0 \quad y_0 \quad z_0] \\ \mathbf{p}_{0.5} &= [x_{0.5} \quad y_{0.5} \quad z_{0.5}] \\ \mathbf{p}_1 &= [x_1 \quad y_1 \quad z_1] \end{aligned} \tag{14.6}$$

where the subscripts indicate the value of the parametric variable at each point.

Now we solve Equations 14.4 for the $a_x, b_x, \ldots, c_z$ coefficients in terms of these points. Thus, for x at $u = 0$, $u = 0.5$, and $u = 1$, we have

$$\begin{aligned} x_0 &= c_x \\ x_{0.5} &= 0.25a_x + 0.5b_x + c_x \\ x_1 &= a_x + b_x + c_x \end{aligned} \tag{14.7}$$

with similar equations for y, and z.

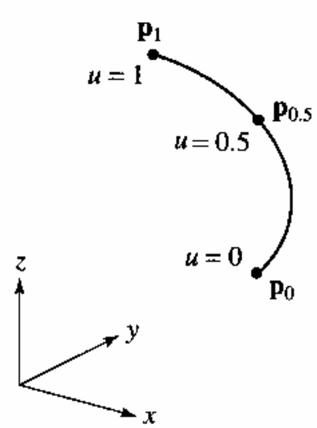


Figure 14.2 *A plane curve defined by three points.*

Next we solve these three equations in three unknowns for a_x, b_x, and c_x, finding

$$\begin{aligned} a_x &= 2x_0 - 4x_{0.5} + 2x_1 \\ b_x &= -3x_0 + 4x_{0.5} - x_1 \\ c_x &= x_0 \end{aligned} \tag{14.8}$$

Substituting this result into Equation 14.4 yields

$$x(u) = (2x_0 - 4x_{0.5} + 2x_1)u^2 + (-3x_0 + 4x_{0.5} - x_1)u + x_0 \tag{14.9}$$

Again, there are equivalent expressions for $y(u)$ and $z(u)$.

We rewrite Equation 14.9 as follows:

$$x(u) = (2u^2 - 3u + 1)x_0 + (-4u^2 + 4u)x_{0.5} + (2u^2 - u)x_1 \tag{14.10}$$

Using this result and equivalent expressions for $y(u)$ and $z(u)$, we combine them into a single vector equation:

$$\mathbf{p}(u) = (2u^2 - 3u + 1)\mathbf{p}_0 + (-4u^2 + 4u)\mathbf{p}_{0.5} + (2u^2 - u)\mathbf{p}_1 \tag{14.11}$$

Equation 14.11 produces the same curve as Equation 14.5. The curve will always lie in a plane no matter what three points we choose. Furthermore, it is interesting to note that the point $\mathbf{p}_{0.5}$ which is on the curve at $u = 0.5$, is not necessarily half way along the length of the curve between $\mathbf{p}_0$ and $\mathbf{p}_1$. We can show this quite convincingly by choosing three points to define a curve such that two of them are relatively close together (Figure 14.3). In fact, if we assign a different value to the parametric variable for the intermediate point, then we obtain different values for the coefficients in Equations 14.8. This, in turn, means that a different curve is produced, although it passes through the same three points.

Equation 14.5 is the *algebraic form* and Equation 14.11 is the *geometric form*. Each of these equations can be written more compactly with matrices. Compactness is not the only advantage to matrix notation. Once a curve is defined in matrix form, we can use the full power of matrix algebra to solve many geometry problems. So now we rewrite

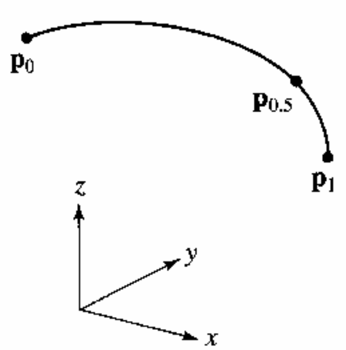


Figure 14.3 *Curve defined by three nonuniformly spaced points.*

Equation 14.5 using the following substitutions:

$$[u^2 \quad u \quad 1]\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{bmatrix} = \mathbf{a}u^2 + \mathbf{b}u + \mathbf{c} \tag{14.12}$$

$$\mathbf{U} = [u^2 \quad u \quad 1] \tag{14.13}$$

$$\mathbf{A} = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}]^T \tag{14.14}$$

and finally, we obtain

$$\mathbf{p}(u) = \mathbf{UA} \tag{14.15}$$

Remember that **A** is really a matrix of vectors, so that

$$\mathbf{A} = \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{bmatrix} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} \tag{14.16}$$

The nine terms on the right are called the *algebraic coefficients*.

Next, we convert Equation 14.11 into matrix form. The right-hand side looks like the product of two matrices: $[(2u^2 - 3u + 1) \quad (-4u^2 + 4u) \quad (2u^2 - u)]$ and $[\mathbf{p}_0 \quad \mathbf{p}_{0.5} \quad \mathbf{p}_1]$. This means that

$$\mathbf{p}(u) = [(2u^2 - 3u + 1) \quad (-4u^2 + 4u) \quad (2u^2 - u)]\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_{0.5}\\ \mathbf{p}_1\end{bmatrix} \tag{14.17}$$

Using the following substitutions:

$$\mathbf{F} = [(2u^2 - 3u + 1) \quad (-4u^2 + 4u) \quad (2u^2 - u)] \tag{14.18}$$

and

$$\mathbf{P} = \begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_{0.5}\\ \mathbf{p}_1\end{bmatrix} = \begin{bmatrix} x_0 & y_0 & z_0 \\ x_{0.5} & y_{0.5} & z_{0.5} \\ x_1 & y_1 & z_1 \end{bmatrix} \tag{14.19}$$

where **P** is the *control point matrix* and the nine terms on the right are its elements or the *geometric coefficients*, we can now write

$$\mathbf{p}(u) = \mathbf{FP} \tag{14.20}$$

This is the matrix version of the geometric form.

Because it is the same curve in algebraic form, $\mathbf{p}(u) = \mathbf{UA}$, or geometric form, $\mathbf{p}(u) = \mathbf{FP}$, we can write

$$\mathbf{FP} = \mathbf{UA} \tag{14.21}$$

The **F** matrix is itself the product of two other matrices:

$$\mathbf{F} = [u^2 \quad u \quad 1]\begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \tag{14.22}$$

The matrix on the left we recognize as **U**, and we can denote the other matrix as

$$\mathbf{M} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \tag{14.23}$$

This means that

$$\mathbf{F} = \mathbf{UM} \tag{14.24}$$

Using this we substitute appropriately to find

$$\mathbf{UMP} = \mathbf{UA} \tag{14.25}$$

Premultiplying each side of this equation by $\mathbf{U}^{-1}$ yields

$$\mathbf{MP} = \mathbf{A} \tag{14.26}$$

This expresses a simple relationship between the algebraic and geometric coefficients

$$\mathbf{A} = \mathbf{MP} \tag{14.27}$$

or

$$\mathbf{P} = \mathbf{M}^{-1}\mathbf{A} \tag{14.28}$$

The matrix **M** is called a *basis transformation matrix*, and **F** is called a *blending function matrix*. There are other basis transformation matrices and blending function matrices, as we shall see in the following sections.

14.3 Space Curves

A space curve is not confined to a plane. It is free to twist through space. To define a space curve we must use parametric functions that are cubic polynomials. For $x(u)$ we write

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x \tag{14.29}$$

with similar expressions for $y(u)$ and $z(u)$. Again, the a, b, c, and d terms are constant coefficients. As we did with Equation 14.5 for a plane curve, we combine the $x(u)$, $y(u)$, and $z(u)$ expressions into a single vector equation:

$$\mathbf{p}(u) = \mathbf{a}u^3 + \mathbf{b}u^2 + \mathbf{c}u + \mathbf{d} \tag{14.30}$$

If $\mathbf{a} = 0$, then this equation is identical to Equation 14.5.

To define a specific curve in space, we use the same approach as we did for a plane curve. This time, though, there are 12 coefficients to be determined. We specify four points through which we want the curve to pass, which provides all the information we need to determine **a**, **b**, **c**, and **d**. But which four points? Two are obvious: $\mathbf{p}(0)$ and $\mathbf{p}(1)$, the end points at $u = 0$ and $u = 1$. For various reasons beyond the scope of this text, it turns out to be advantageous to use two intermediate points that we assign parametric values of $u = \frac{1}{3}$ and $u = \frac{2}{3}$, or $\mathbf{p}(\frac{1}{3})$ and $\mathbf{p}(\frac{2}{3})$. So we now have the four points we need:

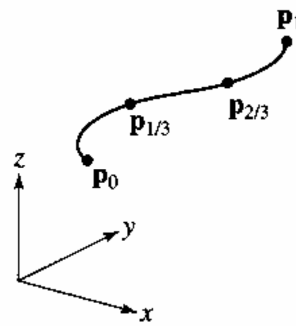


Figure 14.4 *Four points define a cubic space curve.*

$\mathbf{p}(0)$, $\mathbf{p}(\frac{1}{3})$, $\mathbf{p}(\frac{2}{3})$, and $\mathbf{p}(1)$, which we can rewrite as the more convenient $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ (Figure 14.4).

Substituting each of the values of the parametric variable ($u = 0, \frac{1}{3}, \frac{2}{3}, 1$) into Equation 14.29, we obtain the following four equations in four unknowns:

$$\begin{aligned} x_1 &= d_x \\ x_2 &= \frac{1}{27}a_x + \frac{1}{9}b_x + \frac{1}{3}c_x + d_x \\ x_3 &= \frac{8}{27}a_x + \frac{4}{9}b_x + \frac{2}{3}c_x + d_x \\ x_4 &= a_x + b_x + c_x + d_x \end{aligned} \tag{14.31}$$

Now we can express a_x, b_x, c_x, and d_x in terms of x_1, x_2, x_3, and x_4. After doing the necessary algebra, we obtain

$$\begin{aligned} a_x &= -\frac{9}{2}x_1 + \frac{27}{2}x_2 - \frac{27}{2}x_3 + \frac{9}{2}x_4 \\ b_x &= 9x_1 - \frac{45}{2}x_2 + 18x_3 - \frac{9}{2}x_4 \\ c_x &= -\frac{11}{2}x_1 + 9x_2 - \frac{9}{2}x_3 + x_4 \\ d_x &= x_1 \end{aligned} \tag{14.32}$$

We substitute these results into Equation 14.29, producing

$$\begin{aligned} x(u) = &\left(-\frac{9}{2}x_1 + \frac{27}{2}x_2 - \frac{27}{2}x_3 + \frac{9}{2}x_4\right)u^3 \\ &+ \left(9x_1 - \frac{45}{2}x_2 + 18x_3 - \frac{9}{2}x_4\right)u^2 \\ &+ \left(-\frac{11}{2}x_1 + 9x_2 - \frac{9}{2}x_3 + x_4\right)u \\ &+ x_1 \end{aligned} \tag{14.33}$$

All this looks a bit messy right now, but we can put it into a neater, much more compact form. We begin by rewriting Equation 14.33 as follows:

$$\begin{aligned} x(u) = &\left(-\frac{9}{2}u^3 + 9u^2 - \frac{11}{2}u + 1\right) x_1 \\ &+ \left(\frac{27}{2}u^3 - \frac{45}{2}u^2 + 9u\right) x_2 \\ &+ \left(-\frac{27}{2}u^3 + 18u^2 - 9u\right) x_3 \\ &+ \left(\frac{9}{2}u^3 - \frac{9}{2}u^2 + u\right) x_4 \end{aligned} \tag{14.34}$$

Using equivalent expressions for $y(u)$ and $z(u)$, we can summarize them with a single vector equation:

$$\begin{aligned} \mathbf{p}(u) = &\left(-\frac{9}{2}u^3 + 9u^2 - \frac{11}{2}u + 1\right) \mathbf{p}_1 \\ &+ \left(\frac{27}{2}u^3 - \frac{45}{2}u^2 + 9u\right) \mathbf{p}_2 \\ &+ \left(-\frac{27}{2}u^3 + 18u^2 - \frac{9}{2}u\right) \mathbf{p}_3 \\ &+ \left(\frac{9}{2}u^3 - \frac{9}{2}u^2 + u\right) \mathbf{p}_4 \end{aligned} \tag{14.35}$$

This means that, given four points assigned successive values of u (in this case at $u = 0, \frac{1}{3}, \frac{2}{3}, 1$), Equation 14.35 produces a curve that starts at $\mathbf{p}_1$, passes through $\mathbf{p}_2$ and $\mathbf{p}_3$, and ends at $\mathbf{p}_4$.

Now let's take one more step toward a more compact notation. Using the four parametric functions appearing in Equation 14.35, we define a new matrix, $\mathbf{G} = [G_1 \quad G_2 \quad G_3 \quad G_4]$, where

$$\begin{aligned} G_1 &= \left(-\frac{9}{2}u^3 + 9u^2 - \frac{11}{2}u + 1\right) \\ G_2 &= \left(\frac{27}{2}u^3 - \frac{45}{2}u^2 + 9u\right) \\ G_3 &= \left(-\frac{27}{2}u^3 + 18u^2 - \frac{9}{2}u\right) \\ G_4 &= \left(\frac{9}{2}u^3 - \frac{9}{2}u^2 + u\right) \end{aligned} \tag{14.36}$$

and then define a matrix **P** containing the control points, $\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{bmatrix}^T$, so that

$$\mathbf{p}(u) = \mathbf{GP} \tag{14.37}$$

The matrix **G** is the product of two other matrices, **U** and **N**:

$$\mathbf{G} = \mathbf{UN} \tag{14.38}$$

where $\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$ and

$$\mathbf{N} = \begin{bmatrix} -\frac{9}{2} & \frac{27}{2} & -\frac{27}{2} & \frac{9}{2} \\ 9 & -\frac{45}{2} & -\frac{9}{2} & 1 \\ -\frac{11}{2} & 9 & -\frac{9}{2} & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{14.39}$$

(Note that **N** is another example of a basis transformation matrix.)

Now we let

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} \tag{14.40}$$

Using matrices, Equation 14.30 becomes

$$\mathbf{p}(u) = \mathbf{UA} \tag{14.41}$$

which looks a lot like Equation 14.15 for a plane curve, except that we have defined new **U** and **A** matrices. In fact, Equation 24.15 is a special case of the formulation for a space curve.

To convert the information in the **A** matrix into that required for the **P** matrix, we do some simple matrix algebra, using Equations 14.37, 14.38, and 14.41. First we have

$$\mathbf{GP} = \mathbf{UNP} \tag{14.42}$$

and then

$$\mathbf{UA} = \mathbf{UNP} \tag{14.43}$$

or more simply

$$\mathbf{A} = \mathbf{NP} \tag{14.44}$$

14.4 The Tangent Vector

Another way to define a space curve does not use intermediate points. It uses the tangents at each end of a curve, instead. Every point on a curve has a straight line associated with it called the tangent line, which is related to the first derivative of the parametric functions $x(u)$, $y(u)$, and $z(u)$, such as those given by Equation 14.30. Thus

$$\frac{d}{du}x(u), \quad \frac{d}{du}y(u), \quad \text{and} \quad \frac{d}{du}z(u) \tag{14.45}$$

From elementary calculus, we can compute, for example,

$$\frac{dy}{dx} = \frac{dy(u)/du}{dx(u)/du} \tag{14.46}$$

We can treat $dx(u)/du$, $dy(u)/du$, and $dz(u)/du$ as components of a vector along the tangent line to the curve. We call this the *tangent vector*, and define it as

$$\mathbf{p}^u(u) = \left[\frac{d}{du}x(u)\mathbf{i} \quad \frac{d}{du}y(u)\mathbf{j} \quad \frac{d}{du}z(u)\mathbf{k}\right] \tag{14.47}$$

or more simply as

$$\mathbf{p}^u = [x^u \quad y^u \quad z^u] \tag{14.48}$$

(Here the superscript u indicates the first derivative operation with respect to the independent variable u.) This is a very powerful idea, and we will now see how to use it to define a curve.

In the last section, we discussed how to define a curve by specifying four points. Now we have another way to define a curve. We will still use the two end points, but instead of two intermediate points, we will use the tangent vectors at each end to supply the information we need to define a curve (Figure 14.5). By manipulating these tangent vectors, we can control the slope at each end. The set of vectors $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_0^u$, and $\mathbf{p}_1^u$ are called the *boundary conditions*. This method itself is called the *cubic Hermite interpolation*, after C. Hermite (1822–1901) the French mathematician who made significant contributions to our understanding of cubic and quintic polynomials.

We differentiate Equation 14.29 to obtain the x component of the tangent vector:

$$\frac{d}{du}dx(u) = x^u = 3a_x u^2 + 2b_x u + c_x \tag{14.49}$$

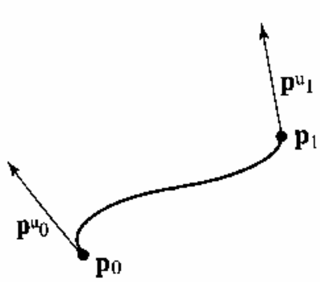


Figure 14.5 *Defining a curve using end points and tangent vectors.*

Evaluating Equations 14.29 and 14.49 at $u = 0$ $u = 1$, yields

$$\begin{aligned} x_0 &= d_x \\ x_1 &= a_x + b_x + c_x + d_x \\ x_0^u &= c_x \\ x_1^u &= 3a_x + 2b_x + c_x \end{aligned} \tag{14.50}$$

Using these four equations in four unknowns, we solve for a_x, b_x, c_x, and d_x in terms of the boundary conditions

$$\begin{aligned} a_x &= 2(x_0 - x_1) + x_0^u + x_1^u \\ b_x &= 3(-x_0 + x_1) - 2x_0^u - x_1^u \\ c_x &= x_0^u \\ d_x &= x_0 \end{aligned} \tag{14.51}$$

Substituting the result into Equation 14.29, yields

$$x(u) = \left(2x_0 - 2x_1 + x_0^u + x_1^u\right)u^3 + \left(-3x_0 + 3x_1 - 2x_0^u - x_1^u\right)u^2 + x_0^u u + x_0 \tag{14.52}$$

Rearranging terms we can rewrite this as

$$\begin{aligned} x(u) = {} & (2u^3 - 3u^2 + 1)x_0 + (-2u^3 + 3u^2)x_1 \\ & + (u^3 - 2u^2 + u)x_0^u + (u^3 - u^2)x_1^u \end{aligned} \tag{14.53}$$

Because $y(u)$ and $z(u)$ have equivalent forms, we can include them by rewriting Equation 14.53 in vector form:

$$\begin{aligned} \mathbf{p}(u) = {} & (2u^3 - 3u^2 + 1)\mathbf{p}_0 + (-2u^3 + 3u^2)\mathbf{p}_1 \\ & + (u^3 - 2u^2 + u)\mathbf{p}_0^u + (u^3 - u^2)\mathbf{p}_1^u \end{aligned} \tag{14.54}$$

To express Equation 14.54 in matrix notation, we first define a blending function matrix $\mathbf{F} = [F_1 \quad F_2 \quad F_3 \quad F_4]$, where

$$\begin{aligned} F_1 &= 2u^3 - 3u^2 + 1 \\ F_2 &= -2u^3 + 3u^2 \\ F_3 &= u^3 - 2u^2 + u \\ F_4 &= u^3 - u^2 \end{aligned} \tag{14.55}$$

These matrix elements are the polynomial coefficients of the vectors in Equation 14.54, which we rewrite as

$$\mathbf{p}(u) = F_1\mathbf{p}_0 + F_2\mathbf{p}_1 + F_3\mathbf{p}_0^u + F_4\mathbf{p}_1^u \tag{14.56}$$

If we assemble the vectors representing the boundary conditions into a matrix $\mathbf{B}$,

$$\mathbf{B} = \left[\mathbf{p}_0 \quad \mathbf{p}_1 \quad \mathbf{p}_0^u \quad \mathbf{p}_1^u\right]^{\mathrm{T}} \tag{14.57}$$

then

$$\mathbf{p}(u) = \mathbf{FB} \tag{14.58}$$

Here, again, we write the matrix **F** as the product of two matrices, **U** and **M**, so that

$$\mathbf{F} = \mathbf{UM} \tag{14.59}$$

where

$$\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \tag{14.60}$$

and

$$\mathbf{M} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{14.61}$$

Rewriting Equation 14.58 using these substitutions, we obtain

$$\mathbf{p}(u) = \mathbf{UMB} \tag{14.62}$$

It is easy to show that the relationship between the algebraic and geometric coefficients for a space curve is the same form as Equation 14.27 for a plane curve. Since

$$\mathbf{p}(u) = \mathbf{UA} \tag{14.63}$$

the relationship between **A** and **B** is, again,

$$\mathbf{A} = \mathbf{MB} \tag{14.64}$$

Consider the four vectors that make up the boundary condition matrix. There is nothing extraordinary about the vectors defining the end points, but what about the two tangent vectors? A tangent vector certainly defines the slope at one end of the curve, but a vector has characteristics of both direction and magnitude. All we need to specify the slope is a unit tangent vector at each end, say $\mathbf{t}_0$ and $\mathbf{t}_1$. But $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{t}_0$, and $\mathbf{t}_1$ supply only 10 of the 12 pieces of information needed to completely determine the curve. So the magnitude of the tangent vector is also necessary and contributes to the shape of the curve. In fact, we can write $\mathbf{p}_0^u$ and $\mathbf{p}_1^u$ as

$$\mathbf{p}_0^u = m_0\mathbf{t}_0 \tag{14.65}$$

and

$$\mathbf{p}_1^u = m_1\mathbf{t}_1 \tag{14.66}$$

Clearly, m_0 and m_1 are the magnitudes of $\mathbf{p}_0^u$ and $\mathbf{p}_1^u$.

Using these relationships, we modify Equation 14.54 as follows:

$$\begin{aligned}\mathbf{p}(u) = {} & (2u^3 - 3u^2 + 1)\mathbf{p}_0 + (-2u^3 + 3u^2)\mathbf{p}_1 \\ & + (u^3 - 2u^2 + u)m_0\mathbf{t}_0 + (u^3 - u^2)m_1\mathbf{t}_1\end{aligned} \tag{14.67}$$

Now we can experiment with a curve (Figure 14.6). Let's hold $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{t}_0$, and $\mathbf{t}_1$ constant and see what happens to the shape of the curve as we vary m_0 and m_1. For simplicity we will consider a curve in the x, y plane. This means that z_0, z_1, z_0^u, and z_1^u

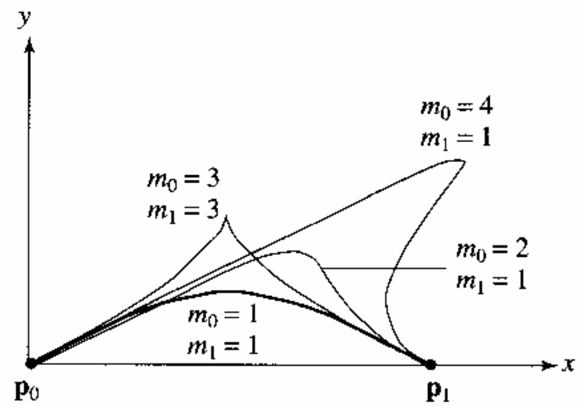


Figure 14.6 *The effect of tangent vector magnitude on curve shape.*

are all equal to zero. The **B** matrix for the curve drawn with the bold line (and with $m_0 = m_1 = 1$) is

$$\mathbf{B} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ m_0\mathbf{t}_0 \\ m_1\mathbf{t}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0.707 & 0.707 & 0 \\ 0.707 & -0.707 & 0 \end{bmatrix} \tag{14.68}$$

Carefully consider this array of 12 elements; they uniquely define the curve. By changing either m_0 or m_1, or both, we can change the shape of the curve. But it is a restricted kind of change, because not only do the end points remain fixed, but the end slopes are also unchanged!

The three curves drawn with light lines in Figure 14.6 show the effects of varying m_0 and m_1. This is a very powerful tool for designing curves, making it possible to join up end to end many curves in a smooth way and still exert some control over the interior shape of each individual curve. For example, as we increase the value of m_0 while holding m_1 fixed, the curve seems to be pushed toward $\mathbf{p}_1$. Keeping m_0 and m_1 equal but increasing their value increases the maximum deflection of the curve from the x axis and increases the curvature at the maximum. (Under some conditions, not necessarily desirable, we can force a loop to form.)

14.5 Blending Functions

The elements of the blending function matrix **F** in Equation 14.55 apply to all parametric cubic curves defined by ends points and tangent vectors at $u = 0$ and $u = 1$. We discussed other blending functions that apply to parametric cubic curves defined by four points. These are the elements of the matrix **G** (Equation 14.36). In fact we can design just about any kind of blending functions, although they may not have many desirable properties.

What blending functions do is "blend" the effects of given geometric constraints, or boundary conditions. Thus, **F** blends the contributions of $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_0^u$, and $\mathbf{p}_1^u$ to create each point on a curve, and **G** blends the contributions of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$. The

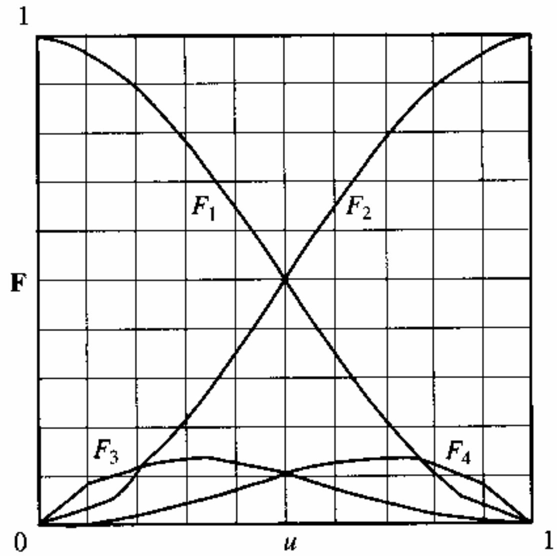
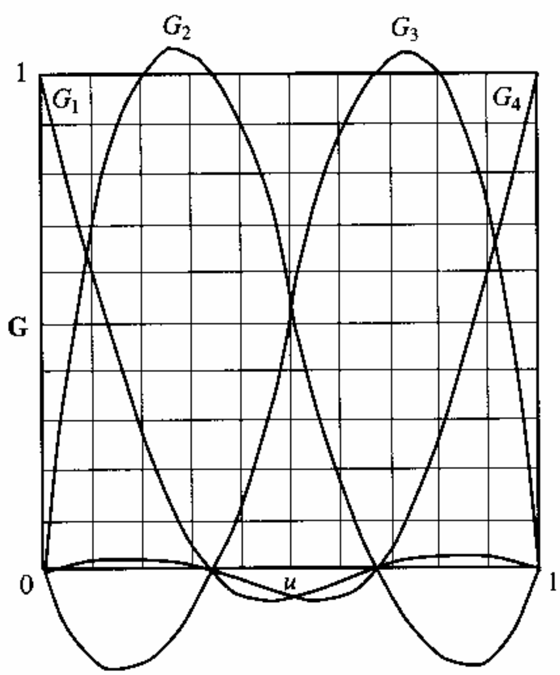


Figure 14.7 **F** *blending functions.*

graphs of F_1, F_2, F_3, and F_4 (Figure 14.7) reveal the mirror-image symmetry between F_1 and F_2, about the line $u = 0.5$. This is also true for F_3 and F_4. We expect this, because there is nothing intrinsically unique about F_1 with respect to F_2, nor about F_3 with respect to F_4. The end point $\mathbf{p}_0$ dominates the shape of the curve for low values of u, through the effect of F_1, while point $\mathbf{p}_1$ acting through F_2 has the greatest influence for values of u near 1.

Next, we consider the graphs of G_1, G_2, G_3, and G_4 (Figure 14.8). Clearly, G_1 and G_4 are symmetrical, as are G_2 and G_3. Note that at $u = 0$, $G_1 = 1$ and G_2, G_3, and

Figure 14.8 **G** *blending functions.*



G_4 equal zero; at $u = \frac{1}{3}$, $G_2 = 1$ and G_1, G_3, and G_4 equal zero; at $u = \frac{2}{3}$, $G_3 = 1$ and G_1, G_2, and G_4 equal zero, and finally at $u = 1$, $G_4 = 1$ and G_1, G_2, and G_3 equal zero.

In each case, the blending functions must have certain properties. These properties are determined primarily by the type of boundary conditions we use to define a curve, and how we may want to alter and control the shape of the curve.

14.6 Approximating a Conic Curve

It is usually possible to substitute a cubic Hermite curve for many other kinds of curves. For example, let us try the conic curves: hyperbola, parabola, ellipse, and circle. Given three points, $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$, there is a conic curve whose tangents at $\mathbf{p}_0$ and $\mathbf{p}_1$ lie along $\mathbf{p}_2 - \mathbf{p}_0$ and $\mathbf{p}_1 - \mathbf{p}_2$, respectively (Figure 14.9). The conic is also tangent to a line parallel to $\mathbf{p}_1 - \mathbf{p}_0$ and offset a distance ρH from that same line. The value of ρ determines the type of conic curve, where

$$\begin{aligned} &\text{Hyperbola: } 0.5 < \rho \leq 1 \\ &\text{Parabola: } \rho = 0.5 \\ &\text{Ellipse: } 0 \leq \rho < 0.5 \end{aligned} \tag{14.69}$$

The complete development and proof of this can be found in advanced textbooks.

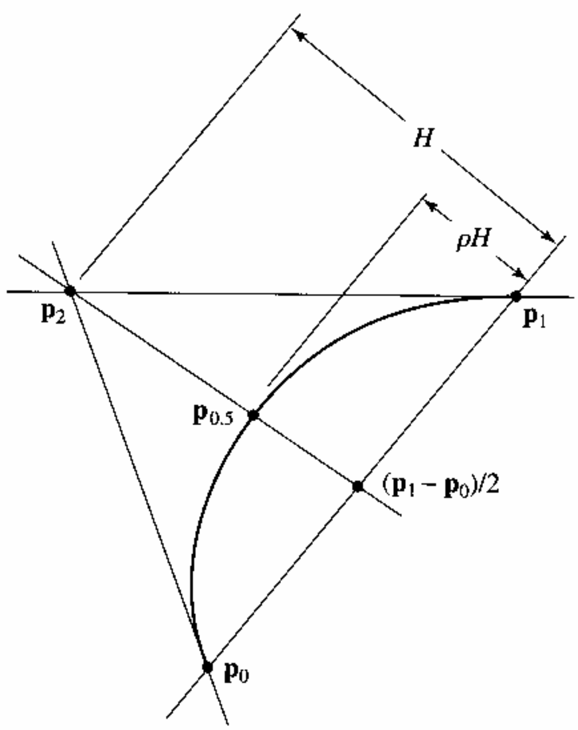


Figure 14.9 *Approximating conic curves.*

The three points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ and ρ define a cubic Hermite curve that is tangent to the lines mentioned earlier and its equation is

$$\mathbf{p}(u) = \mathbf{F}\left[\mathbf{p}_0 \quad \mathbf{p}_1 \quad 4\rho(\mathbf{p}_2 - \mathbf{p}_0) \quad 4\rho(\mathbf{p}_1 - \mathbf{p}_2)\right]^{\mathrm{T}} \tag{14.70}$$

It turns out that this equation exactly fits a parabola and produces good approximations to the hyperbola and ellipse. It is interesting to note that the line connecting the points $\mathbf{p}_2$ and $(\mathbf{p}_1 - \mathbf{p}_0)/2$ intersects the curve at exactly $\mathbf{p}_{0.5}$, and that the tangent vector $\mathbf{p}^u_{0.5}$ is tangent to $\mathbf{p}_1 - \mathbf{p}_0$.

14.7 Reparameterization

We can change the parametric interval in such a way that neither the shape nor the position of the curve is changed. A linear function $v = f(u)$ describes this change. For example, sometimes it is useful to reverse the direction of parameterization of a curve. This is the simplest form of reparameterization. It is quite easy to do. In this example, $v = -u$, where v is the new parametric variable.

Figure 14.10 shows two identically shaped cubic Hermite curves. Their only difference is that they have opposite directions of parameterization. This means that

$$\begin{aligned} \mathbf{q}_0 &= \mathbf{p}_1 \qquad \mathbf{q}^v_0 = -\mathbf{p}^u_1 \\ \mathbf{q}_1 &= \mathbf{p}_0 \qquad \mathbf{q}^v_1 = -\mathbf{p}^u_0 \end{aligned} \tag{14.71}$$

which means we merely interchange $\mathbf{p}_0$ and $\mathbf{p}_1$ and reverse the directions of the tangent vectors.

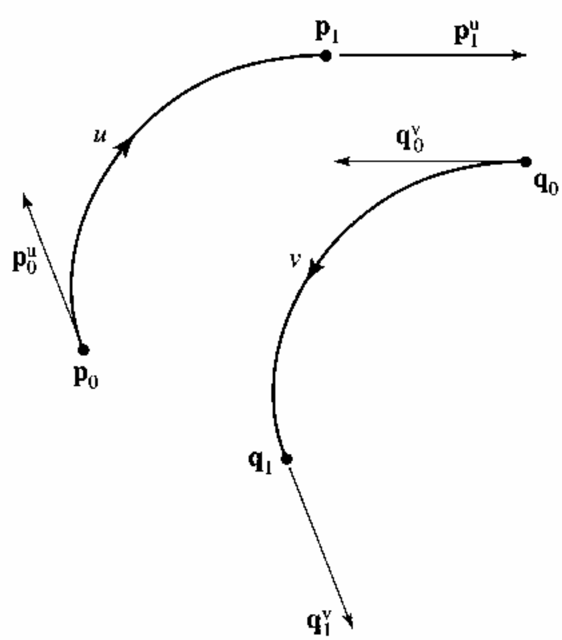


Figure 14.10 *Reversing the direction of parameterization.*

Here is a more general form of reparameterization for cubic Hermite curves. We have a curve that is initially parameterized from u_i to u_j and we must change this so that the parametric variable ranges from v_i to v_j. The initial coefficients are $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{p}_i^u$, and $\mathbf{p}_j^u$, and after reparameterization they are $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_i^v$, and $\mathbf{q}_j^v$.

There is a simple relationship between these sets of coefficients. The end points are related like this: $\mathbf{q}_i = \mathbf{p}_i$ and $\mathbf{q}_j = \mathbf{p}_j$. The tangent vectors require more thought and adjustment. Because they are defined by the first derivative of the parametric basis functions, they are sensitive to the relationship between u and v. A linear relationship is required to preserve the degree of the parametric equations and the directions of the tangent vectors. This means that

$$v = au + b \tag{14.72}$$

Differentiating Equation 14.72, we obtain $dv = adu$. Furthermore, we know that $v_i = au_i + b$ and $v_j = au_j + b$, and we can easily solve for a. Then, since

$$\frac{dx}{du} = a\frac{dx}{dv} \tag{14.73}$$

we find that

$$\mathbf{q}^v = \frac{u_j - u_i}{v_j - v_i}\mathbf{p}^u \tag{14.74}$$

Now we are ready to state the complete relationship between the two sets of geometric coefficients:

$$\begin{aligned} \mathbf{q}_i &= \mathbf{p}_i \qquad & \mathbf{q}_i^v &= \frac{u_j - u_i}{v_j - v_i}\mathbf{p}_i^u \\ \mathbf{q}_j &= \mathbf{p}_j \qquad & \mathbf{q}_j^v &= \frac{u_j - u_i}{v_j - v_i}\mathbf{p}_j^u \end{aligned} \tag{14.75}$$

This tells us that the tangent vector magnitudes must change to accommodate a change in the range of the parametric variable. The magnitudes are scaled by the ratio of the ranges of the parametric variables. The directions of the tangent vectors and the shape and position of the curve are preserved.

14.8 Continuity and Composite Curves

There are many situations in which a single curve is not versatile enough to model a complex shape, and we must join two or more curves together end to end to achieve a design objective. In most cases, but certainly not all, a smooth transition from one curve to the next is a desirable property. We can do this by making the tangent vectors of adjoining curves collinear. However, it is not necessary that their magnitudes are equal, just their direction.

Figure 14.11 shows two curves, $\mathbf{p}(u)$ and $\mathbf{q}(v)$, with tangent continuity. This imposes certain constraints on the geometric coefficients. First, since $\mathbf{p}_1$ and $\mathbf{q}_0$ must

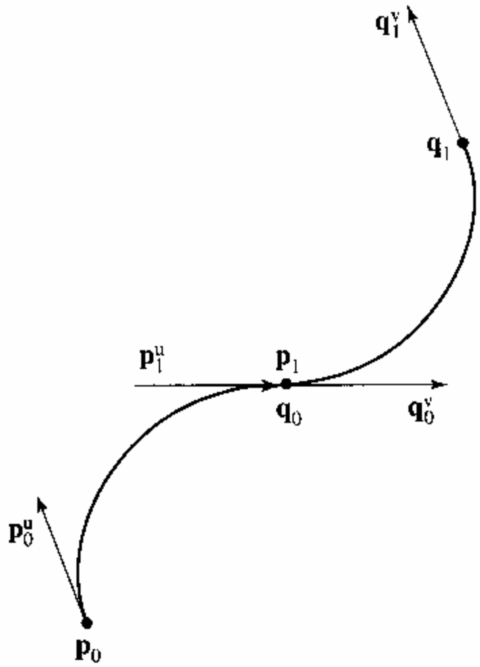


Figure 14.11 *Two curves joined with tangent continuity.*

coincide, we have $\mathbf{q}_0 = \mathbf{p}_1$. Second, the tangent vectors $\mathbf{p}_1^u$ and $\mathbf{q}_0^v$ must be in the same direction, although their magnitudes may differ. This means that $\mathbf{q}_0^v = k\mathbf{p}_1^u$, and the geometric coefficients of $\mathbf{q}(v)$ satisfying these constraints are

$$\mathbf{B}_q = \begin{bmatrix} \mathbf{p}_1 & \mathbf{q}_1 & k\mathbf{p}_1^u & \mathbf{q}_1^v \end{bmatrix}^T \quad (14.76)$$

A composite curve like this has a total of 19 degrees of freedom (compared with 24 for two disjoint curves).

There are various degrees of *parametric continuity* denoted C^n, where n is the degree. C^0 is the minimum degree of continuity between two curves, and indicates that the curves are joined without regard for tangent continuity (i.e., the tangent line is discontinuous at their common point). C^1 indicates first derivative or tangent continuity (discussed earlier), which, of course, presupposes C^0. C^2 indicates second derivative continuity, and is necessary when continuity of curvature at the joint is required. Higher-degree continuity across a joint between two curves is seldom used. There is a related kind of continuity called *geometric continuity*, denoted as G^n, which is not discussed here, but is accessible in more advanced texts.

The notation used in Figure 14.11 and Equation 14.76 is inadequate if more than two or three curves must be used to define a complex curve. A more practical system is suggested here. If n piecewise cubic Hermite curves are joined to form a composite curve of C^1 continuity, we may proceed as follows (Figure 14.12):

1. Label the points consecutively; $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_{n-1}, \mathbf{p}_n$.
2. Define unit tangent vectors, $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_i, \ldots, \mathbf{t}_{n-1}, \mathbf{t}_n$.
3. Define tangent vector magnitudes, $m_{1,0}, m_{1,1}, \ldots, m_{i,0}, m_{i,1}, \ldots, m_{n,0}, m_{n,1}$.

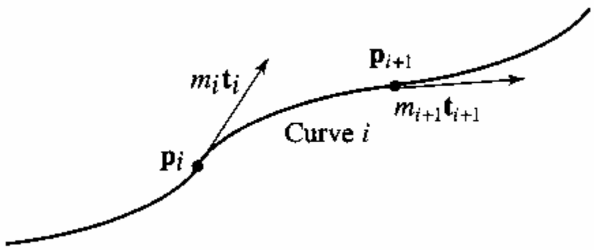


Figure 14.12 *General notation for composite curves.*

Using this notation, the geometric coefficients for curve i are

$$\mathbf{B} = \begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & m_{i,0}\mathbf{t}_i & m_{i,1}\mathbf{t}_{i+1}\end{bmatrix}^T \quad (14.77)$$

Exercises

14.1 Find **a**, **b**, and **c** for each of the curves defined by the following sets of points:

a. $\mathbf{p}_0 = [0 \quad 2 \quad 2]$, $\mathbf{p}_{0.5} = [1 \quad 4 \quad 0]$, $\mathbf{p}_1 = [3 \quad 1 \quad 6]$
b. $\mathbf{p}_0 = [-1 \quad 0 \quad 4]$, $\mathbf{p}_{0.5} = [0 \quad 0 \quad 0]$, $\mathbf{p}_1 = [0 \quad -2 \quad -2]$
c. $\mathbf{p}_0 = [-3 \quad 7 \quad 1]$, $\mathbf{p}_{0.5} = [5 \quad 1 \quad 4]$, $\mathbf{p}_1 = [6 \quad 0 \quad 0]$
d. $\mathbf{p}_0 = [7 \quad 7 \quad 8]$, $\mathbf{p}_{0.5} = [2 \quad 0 \quad 3]$, $\mathbf{p}_1 = [2 \quad -4 \quad 1]$
e. $\mathbf{p}_0 = [0 \quad -1 \quad 2]$, $\mathbf{p}_{0.5} = [-1 \quad -3 \quad 7]$, $\mathbf{p}_1 = [0 \quad 5 \quad 2]$

14.2 Find $\mathbf{p}_0$, $\mathbf{p}_{0.5}$, and $\mathbf{p}_1$ for each of the curves defined by the following sets of algebraic vectors:

a. $\mathbf{a} = [1 \quad 0 \quad 0]$, $\mathbf{b} = [-3 \quad -3 \quad 0]$, $\mathbf{c} = [3 \quad 0 \quad 0]$
b. $\mathbf{a} = [6 \quad 9 \quad 8]$, $\mathbf{b} = [-8 \quad -2 \quad 4]$, $\mathbf{c} = [-4 \quad 6 \quad 1]$
c. $\mathbf{a} = [8 \quad 1 \quad -1]$, $\mathbf{b} = [5 \quad 4 \quad -5]$, $\mathbf{c} = [-10 \quad 4 \quad -3]$
d. $\mathbf{a} = [10 \quad 6 \quad 6]$, $\mathbf{b} = [-15 \quad -17 \quad -13]$, $\mathbf{c} = [7 \quad 7 \quad 8]$
e. $\mathbf{a} = [-2 \quad -4 \quad 4]$, $\mathbf{b} = [3 \quad 2 \quad -10]$, $\mathbf{c} = [-1 \quad 0 \quad 4]$

14.3 What are the dimensions of the matrices in Equation 14.25? Verify the dimensions of the product.

14.4 Find **A** when $\mathbf{P} = \begin{bmatrix} 0 & 1 & 1 \\ 3 & -2 & 0 \\ 2 & 5 & -4 \end{bmatrix}$

14.5 Find $\mathbf{M}^{-1}$

14.6 Describe the curve that results if $\mathbf{p}_0$, $\mathbf{p}_{0.5}$, and $\mathbf{p}_1$ are collinear.

14.7 Give the general geometric coefficients of a curve that lies in the x, y plane.

14.8 Give the general geometric coefficients of a curve that lies in the $y = -3$ plane.

14.9 Compute G_1, G_2, G_3, and G_4 at $u = 0$.

14.10 Compute G_1, G_2, G_3, and G_4 at $u = 1$.

14.11 Compute G_1, G_2, G_3, and G_4 at $u = \frac{1}{3}$.

14.12 Compute G_1, G_2, G_3, and G_4 at $u = \frac{2}{3}$.

14.13 What general conditions must be imposed on the four control points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ to produce a curve that lies in the x, y plane?

14.14 Compute dy/dx for the following functions:

a. $y = 4x^2$

b. $y = x + 3$

c. $y = x^3 - 3x + 1$

d. $y = x^2 + 2x + 1$

e. $y = 2x^4 + x^3 + 3$

14.15 Find the coordinates of the point of zero slope for each of the curves defined in Exercise 14.14.

14.16 Compute m_0 and $\mathbf{t}_0$ for the following tangent vectors:

a. $\mathbf{p}_0^u = [3 \quad -1 \quad 6]$

b. $\mathbf{p}_0^u = [0 \quad 2 \quad 0]$

c. $\mathbf{p}_0^u = [1 \quad 5 \quad -1]$

d. $\mathbf{p}_0^u = [7 \quad 2 \quad 0]$

e. $\mathbf{p}_0^u = [4 \quad 4 \quad -3]$

14.17 Find d^2y/dx^2 for the functions given in Exercise 14.14.

14.18 Given two disjoint (unconnected) curves, $\mathbf{p}(u)$ and $\mathbf{q}(u)$, join $\mathbf{p}_1$ to $\mathbf{q}_0$ with a curve $\mathbf{r}(u)$ such that there is C^1 continuity across the two joints. Write the geometric coefficients $\mathbf{B}_r$ in terms of the coefficients of $\mathbf{p}(u)$ and $\mathbf{q}(u)$.

14.19 How many degrees of freedom (unique coefficients) are required to define the system of three curves created in Exercise 14.18?

14.20 Use the results of Exercise 14.18 to construct a closed composite curve with C^1 continuity by joining $\mathbf{q}_1$ to $\mathbf{p}_0$ with a curve $\mathbf{s}(u)$. Write the geometric coefficients $\mathbf{B}_s$ in terms of the coefficients of $\mathbf{p}(u)$ and $\mathbf{q}(u)$.

14.21 How many degrees of freedom does the closed composite curve of Exercise 14.20 have?