

M. E. Mortenson, *Geometric Modeling*, John Wiley & Sons, New York, 1985, **Chapter 2 Curves, str. 30-33:**

A curve segment is a point-bounded collection of points whose coordinates are given by continuous, one-parameter, single-valued mathematical functions of the form:

$$x = x(u) \quad y = y(u) \quad z = z(u) \tag{1}$$

The parametric variable  $u$  is constrained to the interval to the interval  $u \in [0,1]$ , and the positive sense on a curve is the sense in which  $u$  increases. The curve is point bounded because it has two definite end points, one at  $u = 0$  and the other at  $u = 1$ .

We treat the coordinates of any points on a parametric curve as the components of a vector  $\mathbf{p}(u)$ . Fig. 2.1 illustrates this and other important vector elements. Note that ordinarily the tangent vectors will not be drawn to scale. Here,  $\mathbf{p}(u)$  is the vector to the point  $x(u), y(u), z(u)$ , and  $\mathbf{p}''(u)$  is the tangent vector to the curve at the same point. It is found by differentiating  $\mathbf{p}(u)$ ; thus,

$$\mathbf{p}''(u) = \frac{d\mathbf{p}(u)}{du^2} \tag{2}$$

and the vector components are:

$$x''(u) = \frac{dx(u)}{du^2} \quad y''(u) = \frac{dy(u)}{du^2} \quad z''(u) = \frac{dz(u)}{du^2} \tag{3}$$

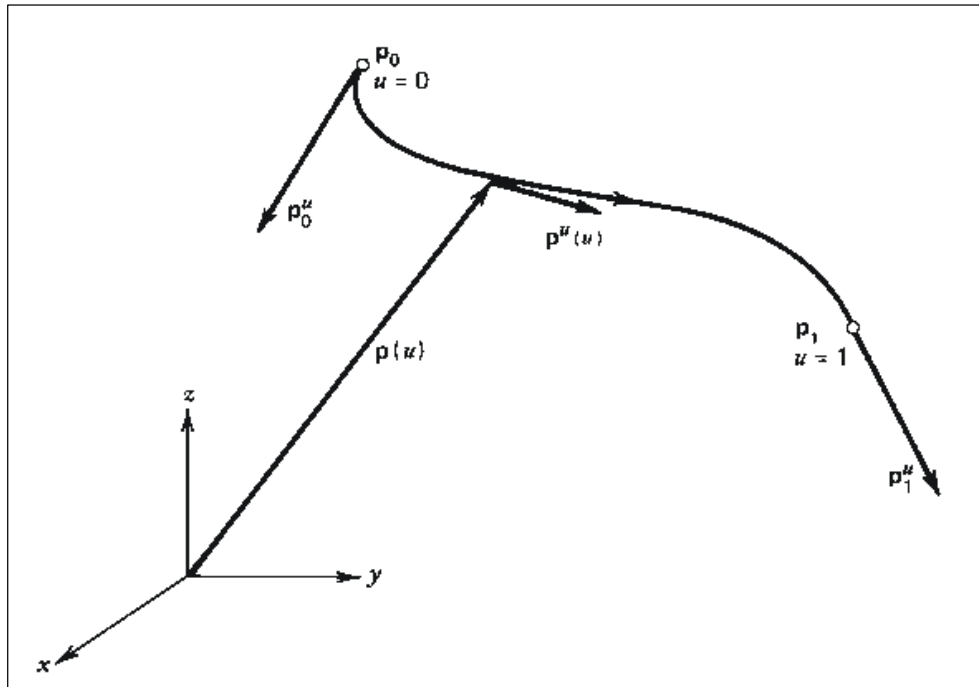


Fig.1

Vector elements of a parametric curve

These are the parametric derivatives. The relationship between the parametric derivatives and the ordinary derivatives of Cartesian space is:

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} \quad (4)$$

and similarly for  $dy/dz$  and  $dz/dx$ . We omit the functional notation when such a reference is obvious from the context, so that, for example:

$$\mathbf{p} = \mathbf{p}(u) \quad \mathbf{p}' = \mathbf{p}'(u) \quad x'' = x''(u) \quad (5)$$

Referring to Figure 2.1 again, the vectors  $\mathbf{p}(0)$ ,  $\mathbf{p}'(0)$ ,  $\mathbf{p}(1)$ , and  $\mathbf{p}'(1)$  are the boundary conditions. The curve is bounded by the point  $\mathbf{p}(0)$  at  $u = 0$  and by the point  $\mathbf{p}(1)$  at  $u = 1$ .

So far, we have presumed that  $\mathbf{p}(u)$  is a three-component vector describing a curve in ordinary three-dimensional Cartesian space; however, it is not restricted to three components. Thus, in general, we have:

$$\mathbf{p}(u) = [x_1(u), x_2(u), \dots, x_i(u), \dots, x_n(u)]$$

for an  $n$ -components curve in an  $n$ -dimensional space.

We know there are ways of representing a curve analytically other than with parametric equations. For example, one equation in  $x, y, z$  represents a surface, and two independent simultaneous equations in  $x, y, z$ , say,

$$F(x, y, z) = 0 \quad G(x, y, z) = 0 \quad (2.10)$$

represent the intersection of two surfaces, which is locally a curve. They are called the implicit equations of a curve. A curve defined this way is inherently unbounded; however, only a bounded part of it may be of interest.

If we solve the implicit equations for two of the variables in terms of the third, for  $y$  and  $z$  in terms of  $x$ , then the results are:

$$y = y(x) \quad z = z(x) \quad (2.11)$$

These equations represent the same curve as Eqs. 2.10, and they, or the equations similarly expressing any two of the coordinates of a variable point on the curve as functions of the third coordinate, are the explicit equations of the curve. Each of Eqs. 2.11 separately represent a cylinder projecting the curve onto one of the principal planes, so these equations are a special form of Eqs. 2.10 for which the two surfaces are projecting cylinders.

If we solve the first of the three parametric Eqs. 2.1 of a curve for  $u$  as a function of  $x$ ,  $u(x)$  and substitute the result into the two remaining expressions, we then obtain the explicit Eqs. 2.11 of the curve. From one point of view, these explicit equations, when supplemented by the identity  $x = x$ , are also parametric equations of the curve:

$$x = x \quad y = y(x) \quad z = z(x) \quad (2.12)$$

The parameter now is the coordinate  $x$ .

The difficulty with this approach is the obviously unacceptable limitation on the range of  $x$ , since the parametric variable must be normalized to the interval  $x \in [0,1]$  by our established convention. We easily resolve the problem by introducing a parametric function of the form:

$$u = \frac{x - x_0}{x_1 - x_0} \quad (2.13)$$

This allows an explicit range of  $x$  from  $x_0$  to  $x_1$ , provided  $u \in [0,1]$ , through the parametric functions  $x(u)$ , where:

$$x = x(u) = x_0 + (x_1 - x_0)u \quad (2.14)$$

This satisfies the normalization constraint on the parametric variable without compromising the range of  $x$ . If we substitute this relationship into Eqs. 2.12, we obtain:

$$x = x(u) \quad y = y[x(u)] \quad z = z[x(u)] \quad (2.15)$$

which simplifies to:

$$x = x(u) \quad y = y(u) \quad z = z(u) \quad (2.16)$$

Clearly, these are the parametric forms introduced in Eqs. 2.1. This means that we can convert a large class of explicit functions into a parametric form.

Bezier (1972) correctly identified the fundamental property of parametric curves: Their shape depends on only the relative position of the points defining their characteristic vectors and is independent of the position of the total set of points with respect to the coordinate system in use. This is an essential characteristic for many applications, such as CAD/CAM modeling. In general, to transform an axis-dependent curve, we must first compute the coordinates of every point required in the original system, then transform each into the new system. For axis-independent curves, it is sufficient to transform the points defining the characteristic vectors from one system to another. Matrix formulation greatly simplifies these operations, as we will see in later sections.

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## 2.1 Algebraic and Geometric Form

The algebraic form of a parametric cubic (pc) curve segment is given by the following three polynomials:

$$\begin{aligned} x(u) &= a_{3x}u^3 + a_{2x}u^2 + a_{1x}u + a_{0x} \\ y(u) &= a_{3y}u^3 + a_{2y}u^2 + a_{1y}u + a_{0y} \\ z(u) &= a_{3z}u^3 + a_{2z}u^2 + a_{1z}u + a_{0z} \end{aligned} \quad (2.17)$$

The parameter  $u$ , the independent variable, is restricted by definition to values in the interval 0 to 1, inclusive. This restriction makes the curve segment bounded.